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A DISTRIBUTION-FREE TEST FOR EXTREMES WITH RADAR APPLICATIONS IN ECM AND ECCM

Defines a statistical test with the property that false-alarm probability can be controlled

J. M. Moser **Research and Development** **20 August 1969**

PROBLEM

Obtain optimum realizable automatic detection methods for future Fleet use. Specifically, develop distribution-free radar detectors for antijamming properties. For this purpose, define and investigate the properties of a statistic which is used to determine whether a sample has extreme values. Investigate whether this statistic has the property of achieving a preassigned probability of falsely rejecting the null hypothesis. Compare the properties of this new statistic to those of the well known Mann-Whitney-Wilcoxon U-statistic and suggest some radar applications.

RESULTS

1. A distribution-free detector (statistical test) is defined. Detection is based upon the sensing of extreme values of the radar signal.

2. The test is compared to the Mann-Whitney-Wilcoxon test. It is found that the new detector, which is distribution-free with respect to the class of distributions that are symmetric about zero, can achieve closely a preassigned probability of falsely rejecting the null hypothesis when it is true provided the size of the sample is large enough. Also, if the sizes of two independent samples are sufficiently large, the detection procedure using the Mann-Whitney-Wilcoxon U-statistic yields the same probability results as those of the new detector using one sample.

3. The detection procedure seems to be poor in one radar application (against single targets) but relatively good in another (against multiple targets).

ADMINISTRATIVE INFORMATION

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INTRODUCTION

In most of the signal detection problems treated in the literature, the physical and statistical characteristics of the signal and noise are well enough known that the functional forms of their distributions can be stated and parameters of the distributions can be specified. Frequently, though, it is possible only to make general assumptions concerning the forms of the distribution functions of noise and signal. We must then find a statistical testing procedure which can distinguish between the signal-present case and the noise-only case on the basis of that small amount of information.

Scope of Report

In this report we discuss the problem of signal detection in a sample of size n under the assumption that signals arise from a stochastically larger population. Such a problem occurs in the radar ECM environment. The time element in signal detection makes the problem relevant. More specifically, the equivalent statistical problem we wish to discuss is that of deciding whether extreme values in a sample come from the same distribution as the main body of the sample.

In general, a statistical decision procedure has the property that the error of rejecting the null hypothesis when it is true (false-alarm probability) can be controlled. For our consideration this means that we wish to control the error of deciding that extreme values are present when actually no extreme values are present.

We describe a distribution-free procedure for determining the presence of extreme values on the basis of a single sample of size n, such as data from a multiple-range-bin radar on a single pulse. A statistic S, based on a sample of size n, is defined and its properties are investigated, especially the property of controlling the error of incorrectly rejecting the null hypothesis.

It is shown that the statistic S is distribution-free over the class of cumulative continuous distribution functions which are symmetric about zero. A comparison of performance is made with the well known Mann-Whitney-Wilcoxon U-statistic. Some applications are presented.

Preliminaries

Any distribution-free decision that a value is extreme must be based only on comparison with other sample values. For example, we compare the k largest values or the q smallest values with the remaining values and on the basis of this comparison make a decision as to whether or not extreme values are present in the sample. The source of our data is a sample of size n $(x_1, \dots, x_n)$. We restrict ourselves to the problem of detecting large extreme values because of the nature of the applications we wish to make. It is easy to see that if we sample simultaneously from two populations, one of which is stochastically larger than the other, we can expect extreme values to occur

more frequently than if we sample from only one population. Therefore, we formulate our hypotheses as follows:

H_0: (no signal). x_i: $i = 1, ..., n$ has distribution $F(x)$, where $F(x) = 1 - F(-x)$; that is, the distribution of x is symmetric about zero.

H_1: (one or more signals). Some proportion p of the sample has distribution $G(x)$ where $G(x) = F(x-a)$, $a > 0$, and the rest of the quantities x come from the population whose distribution is $F(x)$.

This means that we are sampling either from one population or from two populations, one of which is stochastically larger than the other; that is, $G(x)$ is stochastically larger than $F(x)$; $(F(x) \geq G(x))$.

With the above formulation of the hypotheses we wish to prove the following lemmas.

LEMMA I. Given a random sample of size n, $n = 2m$, where each x_i, $i = 1, ..., n$ has the distribution $F(x)$ of H_0, then $g(y_1, ..., y_{2m}) = g(-y_{2m}, ..., -y_1)$ and the distribution of $-y_1, ..., -y_n$ is the same as $y_{m+1}, ..., y_n$.

Proof: $g(y_1, ..., y_{2m}) = n!\left(f(y_1) \ldots f(y_{2m})\right) = n!\, f(-y_1) \ldots f(-y_{2m}) = g(-y_{2m}, ..., -y_1)$

The second part follows.

LEMMA II. Under H_0,

$$P\left[-y_i > y_{n-j}\right] = \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i},$$ where $y_1, ..., y_n$ are the order statistics and as in Lemma I, $n = 2m$, $i \leq m$, $j \leq m-1$, and $y_i \leq 0$.

Proof: $$P\left[-y_i > y_{n-j}\right] = \frac{n!}{(i-1)!j!(n-i-j-1)!} \int_{-\infty}^{0} \int_{y_i}^{-y_i} \left[F(y_i)\right]^{i-1} \left[F(y_{n-j}) - F(y_i)\right]^{n-j-i-1}$$
$$\left[1-F(y_{n-j})\right]^{j} f(y_i) f(y_{n-j})\, dy_{n-j} dy_i$$

Using integration by parts, we obtain

$$\frac{n!}{(n-i-j)!j!(i-1)!} \int_{-\infty}^{0} \left\{ \left[\left[F(y_i)\right]^{j+i-1} \left[1-2F(y_i)\right]^{n-i-j} + j \int_{y_i}^{-y_i} \left[F(y_{n-j}) - F(y_i)\right]^{n-i-j} \right.\right.$$
$$\left.\left. \left[1-F(y_{n-j})\right]^{j-1} \left[F(y_i)\right]^{i-1} f(y_{n-j})\, dy_{n-j} \right] f(y_i)\, dy_i \right\}$$

Applying integration by parts to the inner integral repeatedly yields

$$\frac{n}{(i-1)!} \int_{-\infty}^{0} \sum_{k=0}^{j} (k+i-1)(k+i-2) \ldots (k+1) \binom{n-1}{k+i-1} \left[F(y_i)\right]^{k+i-1} \left[1-2F(y_i)\right]^{n-i-k}$$
$$f(y_i)\, dy_i$$

Let $z = 1 - 2F\left(y_i\right)$. Then we have

$$\frac{n}{(i-1)!} \sum_{k=0}^{j} \frac{(k+i-1)!}{k!} \binom{n-1}{k+i-1} \int_0^1 \left(\frac{1-z}{2}\right)^{k+i-1} z^{n-i-k} \frac{dz}{2} = \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i}$$

which was to be shown.

It should be noted the probability $P\left[-y_i > y_{n-j}\right]$ is distribution-free with respect to the class of distributions which are symmetric about zero. It is not distribution-free, however, with respect to the class of distributions symmetric about a point other than zero.

LEMMA III. Using the same hypothesis as in Lemma II, except that $i < m$, we have

$$P\left[-y_{i+1} < y_{n-j} < -y_i\right] = \binom{i+j}{i} (1/2)^{i+j+1}$$

Proof: $$P\left[-y_{i+1} < y_{n-j} < -y_i\right] = \frac{n!}{(i-1)!(n-i-j-2)!j!} \int_{-\infty}^{0} \int_0^{-y_i} \int_{-y_{n-j}}^{y_{n-j}}$$

$$\left[F\left(y_n\right)\right]^{i-1} \left[F\left(y_{n-j}\right) - F\left(y_{i+1}\right)\right]^{n-j-i-2} \left[1-F\left(y_{n-j}\right)\right]^j f\left(y_i\right) f\left(y_{i+1}\right) f\left(y_{n-j}\right)$$

$$dy_{i+1}\, dy_{n-j}\, dy_i = \frac{n!}{(i-1)!j!(n-i-j-1)!} \int_{-\infty}^{0} \int_0^{-y_i} \left[F\left(y_i\right)\right]^{i-1} \left[1-F\left(y_{n-j}\right)\right]^j$$

$$\left[2F\left(y_{n-j}\right)-1\right]^{n-i-j-1} f\left(y_i\right) f\left(y_{n-j}\right) dy_{n-j}\, dy_i$$

Using integration by parts gives us the result

$$\frac{n!}{(i-1)!j!(n-i-j)!} \int_{-\infty}^{0} \left\{\left[F\left(y_i\right)\right]^{i+j-1} 1/2 \left[1-2F\left(y_i\right)\right]^{n-i-j} + 1/2 \int_0^{-y_i} \left[F\left(y_i\right)\right]^{i-1}\right.$$

$$\left.\left[2F\left(y_{n-j}\right)-1\right]^{n-i-j} j \left[1-F\left(y_{n-j}\right)\right]^{j-1} f\left(y_{n-j}\right) dy_{n-j} \right\} f\left(y_i\right) dy_i$$

Repeated use of integration by parts produces the result

$$\frac{n}{(i-1)!} \int_{-\infty}^{0} \sum_{k=0}^{j} (k+i-1)(k+i-2)\ldots(k+1) \binom{n-1}{k+i-1} \left[F\left(y_i\right)\right]^{k+i-1}$$

$$(1/2)^{j-k+1} \left[1-2F\left(y_i\right)\right]^{n-i-k} f\left(y_i\right) dy_i$$

We let $z = 1-2F\left(y_i\right)$; using a method described in reference 1* yields

$$\sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{j+i+1} = \binom{i+j}{i} (1/2)^{i+j+1}$$

which was to be shown.

It should again be noted that the probability $P\left[-y_{i+1} < y_{n-j} < -y_i\right]$ is distribution-free with respect to the class of distributions symmetric about zero and no other point. These lemmas are used in *DISTRIBUTION OF S.*

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*See REFERENCES.

If we let A be the event $-y_i > y_{n-j} > -y_{i+1}$; and let B be the event $-y_{i+1} > y_{n-j}$; then $A \cup B$ is the event $-y_i > y_{n-j}$. Since A and B are mutually exclusive, $P[A \cup B] = P[A] + P[B]$ or

$$\sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i} = \sum_{k=0}^{j} \binom{k+i}{k} (1/2)^{k+1+i} + \binom{i+j}{i} (1/2)^{i+j+1}$$

The above is shown analytically.

$$\sum_{k=0}^{j} \binom{k+i}{k} (1/2)^{k+1} = \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+1} + \sum_{k=1}^{j} \binom{k+i-1}{k-1} (1/2)^{k+1}$$

We let $k-1 = y$ in the second summation on the right; then we obtain

$$\sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+1} = \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+1} + \sum_{y=0}^{j-1} \binom{y+i}{y} (1/2)^{k+1}$$

$$= \sum_{k=0}^{i} \binom{k+i-1}{k} (1/2)^{k+1} + \sum_{y=0}^{j} \binom{y+i}{y} (1/2)^{y+2} - \binom{i+j}{j} (1/2)^{j+2}$$

Rearranging terms yields

$$1/2 \sum_{k=0}^{j} \binom{k+i}{k} (1/2)^{k+1} = \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+1} - \binom{i+j}{j} (1/2)^{j+2}$$

We multiply both sides of the last equation by $(1/2)^{i-1}$ to obtain the desired result.

THE STATISTIC S

Lemma II of reference 2 strongly influences the definition of the statistic S which we shall shortly present. To this end, then, we consider an n-dimensional random sample of size one, $x_1, \ldots, x_n$, where each x_i has the distribution $F(x)$ as stated in H_0 and where $n = 2m$. We form the order statistics $y_1, y_2, \ldots, y_n$, and change the algebraic signs of the first m order statistics to get

$$-y_1, -y_2, \ldots, -y_m, y_{m+1}, \ldots, y_n.$$

Let us write the sample as $w_1, \ldots, w_m, y_{m+1}, \ldots, y_n$. If we now order the quantities w and the quantities y, we get an arrangement which may look as follows:

$$y_{m+1} < w_m < w_{m-1} < y_{m+2} < \ldots < y_n.$$

This is just one of the $\frac{n!}{m!m!}$ possible arrangements. We now define S as

$$S = \sum_{i=m+1}^{n} R\left(y_i\right)$$

where $R\left(y_i\right)$ is the rank of y_i in the ordered sample of w and y. It is easily seen that $\frac{m(m+1)}{2} \leq S \leq \frac{m(3m+1)}{2}$. The second sum occurs when all the w are less than all the y, and th first sum occurs when the situation is just reversed. If H_0 is true, we should expect the w to be randomly placed between the y. If the w and y actually alternate, $S = m(m+1)$ or m^2, depending on whether we start with a w or a y. Because of H_1, we should expect S to be larger under H_1 than under H_0. For this reason the decision rule is as follows:

Reject H_0 if $S > \lambda$ where λ is an integer so chosen as to yield a false-alarm probability which is less than or equal to α.

DISTRIBUTION OF S

By definition, S is the sum of ranks – that is, of positive integers and the same value of S can be obtained by adding together different integers.

The definition of S, therefore, demands that we count the number of partitions of the integer ℓ which belong to the range of S; $\frac{m(m+1)}{2} \leq \ell \leq \frac{m(3m+1)}{2}$, with the restriction that the number of components in each partition is m. The generating function for enumerating partitions with unequal parts by number of parts is (page 113, reference 3):

$$G(t,a) = \sum_{k=0}^{\infty} u(t,k)a^k$$

where $u(t,k)$ is the generating function for partitions with k unequal parts. Therefore, let us define $G_{b,n}(t,a)$ as:

$$G_{b,n}(t,a) = \sum_{k=0}^{n-b+1} u(t,k)a^k$$

$G_{b,n}(t,a)$ is the function for enumerating partitions of an integer q with unequal parts by number of parts where the smallest part is greater than or equal to b and the largest part is less than or equal to n. The function $u(t,k)$ is the generating function for partitions

with k unequal parts with the restrictions mentioned for $G_{b,n}(t,a)$. This leads us to the following theorem:

THEOREM I. If in $G_{b,n}(t,a)$ we let $k = m$, we obtain

$$u(t,m) = \frac{\left(t^{m+b-1}-t^{n+1}\right)\left(t^{m+b-2}-t^{n+1}\right)\ldots\left(t^{b}-t^{n+1}\right)}{(1-t)\left(1-t^{2}\right)\ldots\left(1-t^{m}\right)}$$

Proof: $G_{b,n}(t,at) = \left(1+at^{b+1}\right)\ldots\left(1+at^{n+1}\right)$ and $\left(1+at^{n+1}\right) G_{b,n}(t,a)$

$= \left(1+at^{b}\right) G_{b,n}(t,at)$, so that

$$\sum_{m=0}^{n-b+1} u(t,m)a^{m} + \sum_{m=0}^{n-b+1} u(t,m)a^{m+1}t^{n+1} = \sum_{m=0}^{n-b+1} u(t,m)a^{m}t^{m} + \sum_{m=0}^{n-b+1} u(t,m)a^{m+1}t^{m+b}$$

Making a change of the variable of addition in the second and last summations and equating coefficients of a^m results in

$$u(t,m)\left[1-t^{m}\right] = u(t,m-1)\left[t^{m+b-1}-t^{n+1}\right] \tag{1}$$

or

$$u(t,m) = u(t,m-1)\frac{\left(t^{m+b-1}-t^{n+1}\right)}{\left(1-t^{m}\right)} \tag{2}$$

With $u(t,0) = 1$ and repeated iterations of (2), the desired result is produced. Definition: Let $N(b, m, n, r)$ be equal to the coefficient of t^r in the expansion of $u(t,m)$ in powers of t. $N(b, m, n, r)$ is the number of partitions of the integer r into m distinct parts such that all parts are greater than or equal to b and less than or equal to n. It should be noted that N is actually a function of only three distinct variables – b, m, and r – since $n = 2m$. In order to find the distribution of S, we must know the probability of each of the $\binom{n}{m}$ possible arrangements of the combined orderings of the w and y. If we order the combined samples of w and y, the result is that we obtain either

a. A run of w, then a run of y, then a run of w again, etc., ending either in a w or a y;

or

b. A run of y, then a run of w, then a run of y again, etc., ending either in a w or a y.

The following theorem produces the probability of a particular arrangement of w and y.

THEOREM II. (1) Under H_0

$$P\left[\text{condition (b) above}\right] = \binom{3m-k-1}{n-k}^{-1} P\left[y_k < w_m < y_{k+1}\right] = (1/2)^{3m-k}$$

$$= (1/2)^{n-\text{rank}\,(w_m)+1}$$

where y_k, $m+1 \leq k \leq n$ is the last y in the beginning run of y and the rank (w_m) is the rank of (w_m) in the combined sample of w and y.

(2) Under H_0

$$P[\text{condition (a) above}] = \binom{m+k-2}{k-1}^{-1} P\left|w_r < y_{m+1} < w_{k-1}\right|$$

$$= (1/2)^{m+k-1} = (1/2)^{n-\text{rank}\,(y_{m+1})+1}$$

where w_k, $1 \leq k \leq m$ is the last of the w in the beginning run of w and rank $\left(y_{m+1}\right)$ is the rank of y_{m+1} in the combined sample of w and y.

Proof: The proof is given for the case where $n = 6$ and $m = 3$. We then have

$$y_1, y_2, y_3, y_4, y_5, y_6$$
$$-y_1, -y_2, -y_3, y_4, y_5, y_6$$
$$w_1, w_2, w_3, y_4, y_5, y_6$$

The number of ordered arrangements of the w and y is 20, and they are:

$$w_3 < w_2 < w_1 < y_4 < y_5 < y_6$$
$$w_3 < w_2 < y_4 < w_1 < y_5 < y_6$$
$$w_3 < w_2 < y_4 < y_5 < w_1 < y_6$$
$$w_3 < w_2 < y_4 < y_5 < y_6 < w_1$$
$$w_3 < y_4 < w_2 < w_1 < y_5 < y_6$$
$$w_3 < y_4 < w_2 < y_5 < w_1 < y_6$$
$$w_3 < y_4 < w_2 < y_5 < y_6 < w_1$$
$$w_3 < y_4 < y_5 < w_2 < w_1 < y_6$$
$$w_3 < y_4 < y_5 < w_2 < y_6 < w_1$$
$$w_3 < y_4 < y_5 < y_6 < w_2 < w_1$$

Now interchanging w_3 with y_4, w_2 with y_5, and w_1 with y_6 yields the other 10 ordered arrangements. We now consider arrangements which satisfy condition (a) and for an example consider

$$w_3 < y_4 < w_2 < w_1 < y_5 < y_6$$
$$w_3 < y_4 < w_2 < y_5 < w_1 < y_6$$
$$w_3 < y_4 < w_2 < y_5 < y_6 < w_1$$
$$w_3 < y_4 < y_5 < w_2 < w_1 < y_6$$
$$w_3 < y_4 < y_5 < w_2 < y_6 < w_1$$
$$w_3 < y_4 < y_5 < y_6 < w_2 < w_1$$

Since these arrangements exhaust the ways in which y_4 can lie between w_3 and w_2, and since each arrangement is mutually exclusive of the others, it is clear that

$$P\left|w_3 < y_4 < w_2 < w_1 < y_5 < y_6\right| + \ldots + P\left|w_3 < y_4 < y_5 < y_6 < w_2 < w_1\right|$$

$$= P\left|w_3 < y_4 < w_2\right| = \binom{2+2}{2} (1/2)^{2+2+1} \text{ from Lemma III.}$$

Next we wish to show that the above arrangements are equiprobable.

$P\left[w_3 < y_4 < y_5 < w_2 < y_6 < w_1\right] = P\left[-y_3 < y_4 < y_5 < -y_2 < y_6 < -y_1\right]$

$$= 6! \int_{-\infty}^{0} \int_{0}^{-y_1} \int_{-y_6}^{0} \int_{0}^{-y_2} \int_{0}^{y_5} \int_{-y_4}^{y_4} f(y_3)f(y_4)f(y_5)f(y_2)f(y_6)f(y_1)\,dy_3dy_4dy_5dy_2dy_6dy_1$$

$$= 6! \int_{-\infty}^{0} \int_{0}^{-y_1} \int_{-y_6}^{0} \int_{0}^{-y_2} \int_{0}^{y_5} \left[2F(y_4)-1\right] f(y_4)f(y_5)f(y_2)f(y_6)f(y_1)\,dy_4dy_5dy_2dy_6dy_1,$$

since $F(y) = 1-F(-y)$,

$$= 6! \int_{-\infty}^{0} \int_{0}^{-y_1} \int_{-y_6}^{0} \int_{0}^{-y_2} 1/2 \frac{[2F(y_5)-1]^2}{2} f(y_5)f(y_2)f(y_6)f(y_1)\,dy_5dy_2dy_6dy_1$$

$$= 6! \int_{-\infty}^{0} \int_{0}^{-y_1} \int_{-y_6}^{0} 1/4 \frac{[1-2F(y_2)]^3}{3!} f(y_2)f(y_6)f(y_1)\,dy_2dy_6dy_1$$

$$= 6! \int_{-\infty}^{0} \int_{0}^{-y_1} 1/8 \frac{[2F(y_6)-1]^4}{4!} f(y_6)f(y_1)\,dy_6dy_1$$

$$= 6! \int_{-\infty}^{0} 1/16 \frac{[1-2F(y_1)]^5}{5!} f(y_1)\,dy_1$$

$$= 6! \frac{1}{32} \cdot \frac{1}{6!} = \frac{1}{32}$$

$P\left[w_3 < y_4 < w_2 < y_5 < y_6 < w_1\right] = P\left[-y_3 < y_4 < -y_2 < y_5 < y_6 < -y_1\right]$

$$= 6! \int_{-\infty}^{0} \int_{0}^{-y_1} \int_{0}^{y_6} \int_{-y_5}^{0} \int_{0}^{-y_2} \int_{-y_4}^{y_4} f(y_3)f(y_4)f(y_2)f(y_5)f(y_6)f(y_1)\,dy_3dy_4dy_2dy_5dy_6dy_1$$

$$= \frac{1}{32}$$

$P\left[w_3 < y_4 < w_2 < w_1 < y_5 < y_6\right] = P\left[-y_3 < y_4 < -y_2 < -y_1 < y_5 < -y_6\right]$

$$= 6! \int_{0}^{\infty} \int_{0}^{y_6} \int_{0}^{0} \int_{-y_5}^{0} \int_{y_1}^{-y_2} \int_{-y_4}^{y_4} f(y_3)f(y_4)f(y_2)f(y_1)f(y_5)f(y_6)\,dy_3dy_4dy_2dy_1dy_5dy_6$$

$$= \frac{1}{32}$$

It is easily shown that the other three probabilities all are equal to $\frac{1}{32}$. Therefore, we obtain the general result

$$P\left[\text{condition (a) above}\right] = \binom{m+k-2}{k-1}^{-1} P\left[w_k < y_{m+1} < w_{k-1}\right] = (1/2)^{m+k-1}$$

The example immediately gives us the result

$$(1/2)^{m+k-1} = (1/2)^{n-\text{rank}(y_{m+1})+1}$$

In a similar manner we show that

$$P\left[\text{condition (b) above}\right] = \binom{3m-k-1}{n-k}^{-1} P\left[y_k < w_m < y_{k+1}\right]$$

$$= (1/2)^{3m-k} = (1/2)^{n\text{-rank}(w_m)+1}$$

Corollary I

$$P\left[-y_m < \ldots < -y_1 < y_{m+1} < \ldots < y_n\right] = P\left[y_{m+1} < \ldots < y_n < -y_m < \ldots < -y_n\right] = (1/2)^m$$

The proof can also be easily obtained with Lemma II. The distribution of S is obtained with the next theorem:

THEOREM III. Under H_0

$$\begin{aligned} P[s=r] &= P[s=r \mid \text{condition (a) is true}] \\ &+ P[s=n \mid \text{condition (b) is true}] \\ &= [N(1,m,n,r) - N(2,m,n,r)]\ (1/2)^{3m-1} \\ &+ \sum_{b=2}^{m+1} [N(b,m,n,r) - N(b+1,m,n,r)]\ (1/2)^{m+b-2} \end{aligned}$$

where $N(m+2,m,n,r) = 0$

Proof: The proof of the theorem follows from the definition of $N(b,m,n,r)$ and Theorem II.

Definition: λ is the critical integer; that is, $P[S \geq \lambda] \leq \alpha$

If $\alpha = 10^{-6}$, we need to consider a sample of dimension greater than or equal to 40 to obtain the desired critical region. The distribution of S for $n = 2, 4, 6, 8$ appears in table 1. The mean and variance of S are obtained in the next two theorems.

THEOREM IV

$$E[S] = \frac{m(3m+1)}{2} - \sum_{i=1}^{m} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i}$$

Proof: Let $U_{i,\,n-j} = \begin{cases} 1 \text{ if } y_i > y_{n-j} \\ 0 \text{ otherwise} \end{cases}$. Then $\sum_{i,j} U_{i,\,n-j}$ equals the number of times the negative y (that is, the w) exceed the positive y, so that $S = \frac{m(3m+1)}{2} - \sum_{i,j} U_{i,\,n-j}$ and

TABLE 1. DISTRIBUTION OF S.

		r	$P[S=r]$
$m=1$	$n=2$		
		1	1/2
		2	1/2
$m=2$	$n=4$	3	1/4
		4	1/8
		5	1/4
		6	1/8
		7	1/4
$m=3$	$n=6$	6	1/8
		7	1/16
		8	3/32
		9	1/8
		10	3/32
		11	3/32
		12	1/8
		13	3/32
		14	1/16
		15	1/8
$m=4$	$n=8$	r	$P[S=r]$
		10	1/16
		11	1/32
		12	3/64
		13	7/128
		14	5/64
		15	7/128
		16	9/128
		17	1/16
		18	5/64
		19	1/16
		20	9/128
		21	7/128
		22	5/64
		23	7/128
		24	3/64
		25	1/32
		26	1/16

$$E[S] = \frac{m(3m+1)}{2} - \sum_{i,j} F\left[U_{i,\,n-j}\right] = \frac{m(3m+1)}{2} - \sum_{i,j} P\left[-y_i > y_{n-j}\right]$$

$$= \frac{m(3m+1)}{2} - \sum_{i=1}^{m} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i}$$

THEOREM V

$$V[S] = \sum_{i=1}^{m} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i} - \left\{ \sum_{i=1}^{m} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i} \right\}^2$$

$$+ 2 \sum_{i=1}^{m} \sum_{k=2}^{m-1} \sum_{j=0}^{k-1} \sum_{q=0}^{j} \binom{q+i-1}{q} (1/2)^{q+i}$$

$$+ 2 \sum_{j=0}^{m-1} \sum_{k=3}^{m-1} \sum_{i=1}^{k-1} \sum_{q=0}^{j} \binom{q+i-1}{q} (1/2)^{q+i}$$

$$+ 2 \sum_{\ell=3}^{m} \sum_{\ell=1}^{i-1} \sum_{k-2}^{m-1} \sum_{j-0}^{k-1} \sum_{q=0}^{j} \binom{q+i-1}{q} (1/2)^{q+i}$$

$$+ 2 \sum_{\ell=3}^{m} \sum_{i=1}^{\ell-1} \sum_{k=2}^{m-1} \sum_{j=0}^{k-1} \left\{ \left(\sum_{n=0}^{\ell-1-i} \binom{j+\ell-1-r}{j} (1/2)^{j+\ell-r} \right) \left(\sum_{q=0}^{k} \binom{q+\ell-1}{q} (1/2)^{q+\ell} \right) \right.$$

$$\left. + \sum_{q=0}^{j} \binom{q+i-1}{q} (1/2)^{q+\ell} \right\}$$

Proof: Since S is of the form $b + w$, b a constant and w a random variable, and since $V(b + w) = V[w]$, it is easily seen that

$$V[S] = V\left[\sum_{i,j} U_{i,\,n-j}\right] = E\left(\sum_{i,j} U_{i,n-j}\right)^2 - \left(\sum_{i,j} E\left(U_{i,\,n-j}\right)\right)^2$$

$$= \sum_{i,j} E\left[U_{i,n-j}^{2}{}^{2}\right] + \sum_{\substack{i\neq k \\ j\neq \ell}\ \text{and}\ \substack{i=k \\ j\neq \ell}\ \text{and}\ \substack{i\neq k \\ j=\ell}} E\left[U_{i,n-j}\, U_{k,n-\ell}\right]$$

$$- \left\{ \sum_{\ell=1}^{m} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i} \right\}^2$$

But, $E[U^2] = E[U]$ so

$$V[S] = \sum_{i=1}^{m} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i} - \left\{ \sum_{i=1}^{m} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i} \right\}^2$$

$$+ \sum E\left[U_{i,\,n-j} U_{k,\,n-\ell}\right]$$

If $i=\ell$, and $n\text{-}j \neq n\text{-}k$, then

$E\left[U_{i,\,n-j}\; U_{i,\,n-k}\right] = P\left[-y_i > y_{n-j} \text{ and } -y_i > y_{n-k}\right] = P\left[-y_i > y_a\right]$ when $a = \max\left[n\text{-}j, n\text{-}k\right]$.

If $n\text{-}j = n\text{-}k$ and $i \neq \ell$, then

$E\left[U_{i,\,n-j}\; U_{\ell,\,n-j}\right] = P\left[-y_b > y_{n-j}\right]$ where $b = \max\left[i, \ell\right]$.

If $i > \ell$ and $n\text{-}j > n\text{-}k$, then

$E\left[U_{i,\,n-j}\; U_{\ell,\,n-k}\right] = P\left[-y_i > y_{n-j}\right]$.

If $i < \ell$ and $n\text{-}j > n\text{-}k$, then

$E\left[U_{i,\,n-j}\; U_{\ell,\,n-k}\right] = P\left[-y < y_{n-j} < -y_i\right]\; P\left[-y > y_{n-k}\right] + P\left[-y > y_{n-j}\right]$.

Substituting the probabilities into the formula produces the desired result.

TWO-SAMPLE COMPARISON

Since the purpose of ordering the n-dimensional sample, then changing the signs of the first m ($n = 2m$) values, was to compare half of the sample (the upper half) with the remaining half (the lower half), it is therefore appropriate to ask how the foregoing development would appear if we actually used two independent m-dimensional samples. We therefore wish to prove the following lemmas.

THEOREM VI. Given $x_{(1)}, \ldots, x_{(m)}$ and $y_{(1)}, \ldots, y_{(m)}$, two independent m-dimensional samples of order statistics from a *cdf* $F(x)$ when $F(x)$ need only be continuous, then

$$P\left[x_{(m-i)} < y_{(m-j)} < x_{(m-i+1)}\right] = \binom{m}{m-i} \int_{-\infty}^{\infty} [F(t)]^{2m-i-j-1}\, [1-F(t)]^{i+j}\, \frac{m!}{(m-1-j)!j!}\, dF(t)$$

$$= \frac{(m-j)\binom{m}{i}\binom{m}{j}}{(2m-i-j)\binom{2m}{i+j}} \quad i < m;\; j < m$$

Proof: With the use of

$$P\left[r_s = k\right] = \binom{n-1}{k-1} \int_{-\infty}^{\infty} F^{k-1}\, (1-F)^{n-k}\, dG$$

where n-1 observations have *cdf* F and one observation has *cdf* G, r_s is the rank of the observation having *cdf* G in the combined sample; the theorem can easily be proved.

THEOREM VII. Given the hypothesis stated in Theorem VI, then

$$P\left[x_{(m-i+1)} > y_{(m-j)}\right] = (m-j+1) \sum_{k=m-i}^{m} \frac{\binom{m}{k}\binom{m}{i-1}}{\binom{n}{k+m-i+1}(k+m-j+1)}$$

where $i \leq m$ and $j \leq m$.

Proof:

$$P\left[x_{(m-j+1)} > y_{(m-j)}\right] = \int_{-\infty}^{\infty} P\left[y_{m-j} < x_{(m-i+1)} \mid x_{(m-j+1)} = x_{(m-i+1)}\right] dP\left[x_{(m-i+1)} \leq x_{(m-i+1)}\right]$$

$$= \int_{-\infty}^{\infty} P\left[y_{m-j} < x_{(m-i+1)}\right] dP\left[x_{(m-i+1)} \leq x_{(m-i+1)}\right]$$

$$= \sum_{k=m-j}^{m} \int_{-\infty}^{\infty} \binom{m}{k} \left[F\left(x_{(m-i+1)}\right)\right]^{k} \left[1-F\left(x_{(m-i+1)}\right)\right]^{m-k}$$

$$\frac{m!}{(m-i)!(i-1)!} \left[F\left(x_{(m-i+1)}\right)\right]^{m-i} \left[1-F\left(x_{(m-i+1)}\right)\right]^{i-1} dF\left(x_{(m-i+1)}\right)$$

$$= \sum_{k=m-j}^{m} \int_{-\infty}^{\infty} \binom{m}{k} \left[F\left(x_{(m-i+1)}\right)\right]^{k+m-i} \left[1-F\left(x_{(m-i+1)}\right)\right]^{m-k+i-1}$$

$$\frac{m!}{(m-i)!(i-1)!} dF\left(x_{(m-i+1)}\right)$$

Let $u = \left(F\ x_{(m-i+1)}\right)$; then we have

$$= \sum_{k=m-j}^{m} (m-i+1) \binom{m}{i-1} \binom{m}{k} \int_{-\infty}^{\infty} u^{k+m-i} (1-u)^{m-k+i-1}\, du$$

$$= (m-i+1) \sum_{k=m-j}^{m} \frac{\binom{m}{i-1}\binom{m}{k}}{\binom{2m}{k+m-i+1}(k+m-i+1)}$$

which was to be shown.

THEOREM VIII. Given the hypothesis stated in Theorem VI, then

$$\lim_{m \to \infty} P\left[x_{(m-i+1)} > y_{(m-j)}\right] = \sum_{k=0}^{j} \binom{k+i-1}{k} (1/2)^{k+i}$$

$$\text{Proof: } \lim_{m\to\infty} \mathrm{P}\left[x_{(m-i+1)} > y_{(m-j)}\right] = \lim_{m\to\infty} \sum_{k=m-j}^{m} \frac{\binom{m}{i-1}\binom{m}{k}(m-i+1)}{\binom{2m}{k+m-i+1}(k+m-i+1)}$$

Let $g = k\text{-}m\text{+}j$; then we have

$$\lim_{m\to\infty} \sum_{i=0}^{j} \frac{\binom{m}{i-1}\binom{m}{g+m-j}(m-i+1)}{\binom{2m}{2m+g-j-i+1}(2m+g-j-i+1)}$$, which, through a method described in

reference 4, results in $\sum_{i=0}^{j} \binom{j-g+i-1}{i-1} (1/2)^{j-g+i}$. Now, letting $j\text{-}1 = w$, we obtain

$\sum_{w=j}^{0} \binom{w+i-1}{w} (1/2)^{w+i} = \sum_{w=0}^{j} \binom{w+i-1}{w} (1/2)^{w+i}$. However, by Lemma II this is

$\mathrm{P}\left[-y_i > y_{n-j}\right]$.

THEOREM IX. Given the hypothesis stated in Theorem VI, then

$$\lim_{m\to\infty} \mathrm{P}\left[x_{(m-i)} < y_{(m-j)} < x_{(m-i+1)}\right] = \binom{i+j}{i} (1/2)^{i+j+1}$$

$$\text{Proof: } \lim_{m\to\infty} \frac{(m-i)\binom{m}{j}\binom{m}{j}}{(2m-i-j)\binom{2m}{i+j}} = \binom{i+j}{i} (1/2)^{i+j+1}$$ which is obtained by using

reference 4. However, by Lemma III this is $\mathrm{P}\left[-y_{i+1} < y_{m-j} < -y_i\right]$. As demonstrated, the distribution of S depends directly on Lemma III and Lemma II, so that for large values of m, the same values for $\mathrm{P}\left[-y_{i+1} < y_{n-j} < -y_i\right]$ and $\mathrm{P}\left[-y_i > y_{n-j}\right]$ are obtained in the one- and two-sample case, and, hence, the procedure of using S leads to the same result. It is known that if we have two independent samples, the statistic S becomes the Mann-Whitney-Wilcoxon statistic, which, as is well known, is asymptotically normally distributed. Because of the great difficulty of obtaining detection probabilities, they are not included in this report.

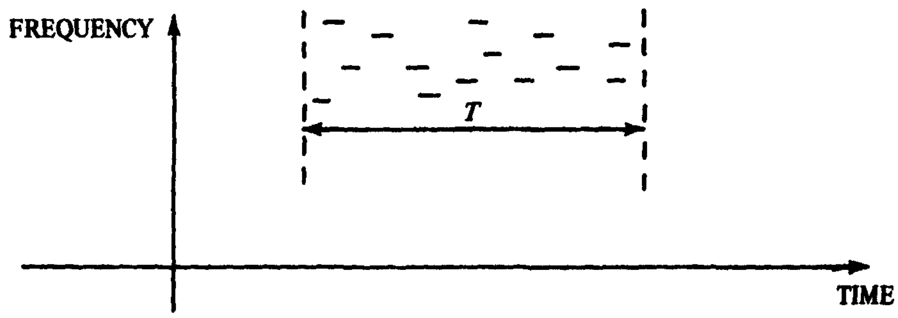
APPLICATIONS

The distributions considered in hypotheses H_0 and H_1 satisfy Lemma I, reference 2, and because of Lemma II, reference 2, a one-sided test was considered. If we are interested in detecting a single enemy object, such as an airplane, using a radar device, we can

do better using the statistics considered in reference 2 than the one considered here, because the statistic S is relatively insensitive to one extreme value in that it does not consider its magnitude. The statistic S is sensitive to enemy objects in the *plural* – to a fleet of gun boats, for example, or a number of dense air targets or hostile troops. It would serve a valuable function therefore in detecting a concentration of the enemy.

A second application is that of intercept of frequency-hopped signals (radar or communications).

Suppose that the message or radar transmission consists of a number of frequency-hopped pulses.



The detection system illustrated uses a bank of narrowband filters, each of bandwidth W. The output of each filter is energy-integrated over time T, the message length or radar transmission length. (This could be continuous integration of a moving-window type or some form of integrate-and-dump – the false-alarm rate depends on the decision rate.) In the presence of a frequency-hopped signal, a substantial portion of the frequency cells are occupied at various times during the signal time T.

The distribution of the integrated energy $V_i(t)$ is given by the chi-square distribution with 2 TW degrees in the noise case (reference 5). TW will be very large, and Urkowitz shows that for large values of TW the normal distribution is a very good approximation to the distribution of $V(t)$. Since the quantities $V_i(t)$ have been normalized by the noise power density for the respective filters, all x_i will have approximately the same variance. Thus, the x_i are identically and symmetrically distributed about zero in the noise case. In the presence of certain types of interference, this will not always be true. However, the statistical test described tends to alarm only when a substantial number of resolution cells are occupied (even at low signal-to-noise ratios), and its false-alarm rate will not increase appreciably when only one or a few cells have interference signals, even very strong signals (fig. 1). Thus, the false-alarm rate will tend to be constant even in the presence of noise plus interference.

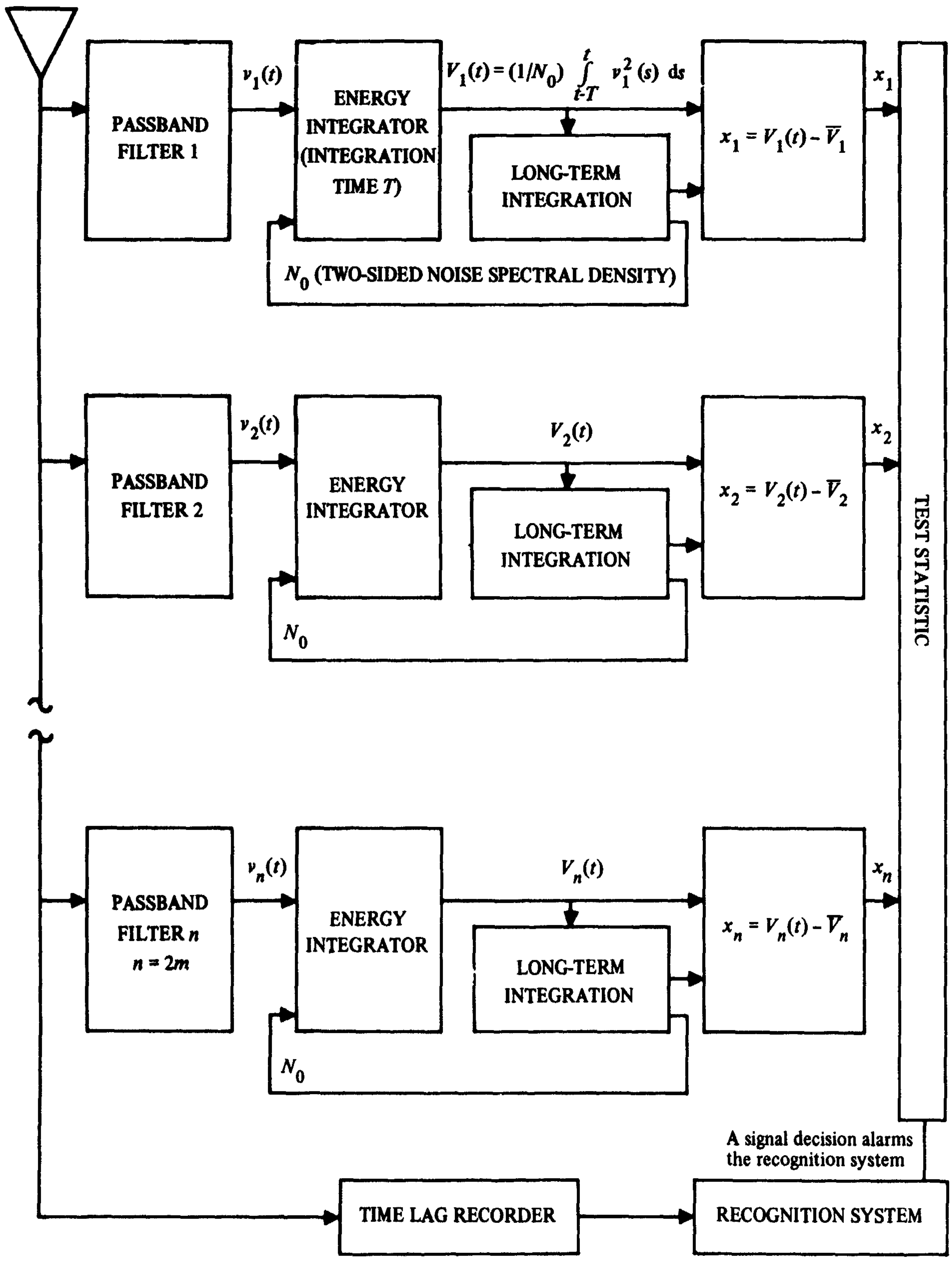


Figure 1. Interceptor detection system.

CONCLUSIONS AND RECOMMENDATIONS

It has been shown that the statistic S has the property that any value of α can be closely obtained provided the dimension of the sample is large enough and the underlying distribution $F(x)$ has the property that $F(x) = 1 - F(-x)$. Also, if we consider two independent samples from a continuous distribution $F(x)$ and the dimension of both samples is large, then the procedure using S leads to the same result in both cases; that is, the case in which $F(x) = 1 - F(-x)$ and one sample is used and the case in which $F(x)$ is continuous and two samples are used. The test procedure using S seems to be poor in one radar application (against single targets) but relatively good in another (against multiple targets). It was also indicated that S can be used to intercept frequency-hopped signals.

It is recommended that before this new detector is implemented some detection probability results be obtained either by analysis or by simulation by Monte Carlo methods, for example.

REFERENCES

1. Feller, W., *An Introduction to Probability Theory and Its Applications,* v. 1, p. 62, Wiley, 1950

2. Naval Electronics Laboratory Center Report 1570, *One-Pulse Signal Detection*, by J.M. Moser and R.E. Simmons, 19 July 1968

3. Riordan, J., *An Introduction to Combinatorial Analysis*, Wiley, 1958

4. Parzen, E., *Modern Probability Theory and Its Applications*, p. 53-54, Wiley, 1960

5. Urkowitz, H., "Energy Detection of Unknown Deterministic Signals," *Institute of Electrical and Electronics Engineers. Proceedings*, v. 55, p. 523-531, April 1967

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13 ABSTRACT

As part of the development of distribution-free detectors (statistical tests) for radar application, the problem of detecting extreme values in a sample is investigated. A rank statistic S is defined and its properties are studied. The statistic is found capable of achieving a preassigned probability of falsely rejecting the null hypothesis when it is true, provided the sample is large enough. False-alarm probability, that is to say, can be controlled. Properties of the statistic are compared to those of the Mann-Whitney-Wilcoxon U-statistic. Radar applications are suggested.

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